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MIXED BOUNDARY VALUE PROBLEMS FOR A CYLINDRICAL SHELL

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Abstract—Two mixed boundary value problems in potential theory for a semi-infinite cylindrical shell are solved. The first is interpreted as a heat conduction problem for an insulated shell containing a circumferential obstruction. The second is the torsion of a cylindrical shell containing a circumferential crack. For both problems the normal derivative of the potential function is taken as zero on the inner and outer shell walls. The boundary conditions at the end of the shell are mixed with respect to the potential function and its normal derivative. The problems are formulated using integral transforms in a manner leading to a singular integral equation which can be solved by numerical means. Intensity factors along the circumference separating the mixed conditions are computed.

1. INTRODUCTION

In this paper two mixed boundary value problems in potential theory for a cylindrical shell geometry are considered: axially symmetric potential theory governed by the equation, $\nabla^2\psi(r, z) = 0$, and axially symmetric torsion governed by the equation, $\nabla^2\psi_1(r, z)\cos\theta = 0$. Here, ∇^2 is the Laplace operator. The problems are such that the normal derivative of the potential function is taken as zero on the shell walls, $r = 1$ and $r = 1 + d = \beta$, and the boundary conditions are mixed with respect to the function and its normal derivative on the end of the shell, $z = 0$, $1 < r < \beta$ (see Fig. 1).

The problems are formulated by means of integral transforms in a form that contains solutions for an infinite cylinder and for a cylindrical cavity. By suitable definition of integral transforms analogous to that of Erdogan[1], the considered problems are reduced to singular integral equations. The equations, prepared for solution by the technique of Erdogan, Gupta and Cook[2], are solved for the pertinent physical quantity, the intensity factor.

It is noted that the kernels of the integral equations are infinite integrals which have a rather slow rate of convergence. The convergence can be improved by subtracting the slowly convergent parts of the integrand which are the leading terms in its asymptotic expansion. These slowly convergent terms can be evaluated in closed form, thereby leading to a rapidly converging infinite integral which can be evaluated numerically along with a closed form expression.

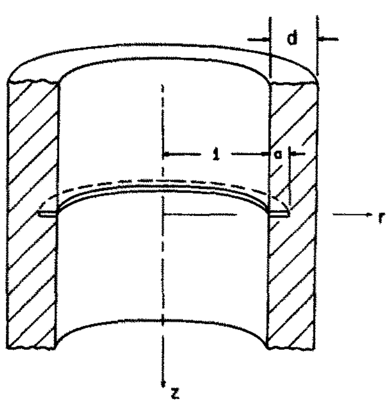

Fig. 1.

2. AXIALLY SYMMETRIC POTENTIAL THEORY PROBLEM

The axially symmetric potential theory problem for the cylindrical shell geometry shown in Fig. 1 is considered. The inner radius of the shell is $r = 1$ and the outer radius is $r = 1 + d = \beta$ where d is the shell thickness and r, θ, and z are cylindrical coordinates and the z axis is the shell axis. We also define $\gamma = 1 + a$ where a is the radial crack length. The boundary conditions are

$$\partial\psi/\partial r = 0; \quad r = 1, \beta; \quad 0 \le z < \infty \tag{1a,b}$$

$$\partial\psi/\partial z = -p(r); \quad z = 0; \quad 1 \le r \le \gamma \tag{2}$$

$$\psi = 0; \quad z = 0; \quad \gamma \le r \le \beta \tag{3}$$

where $\psi = \psi(r, z)$ and

$$\nabla^2\psi = 0, \ -\infty < z < \infty, \ 1 \le r \le \beta. \tag{4}$$

A suitable choice for $\psi(r, z)$ that satisfies eqn (4) is given by

$$\psi(r, z) = \int_0^\infty A(\xi)e^{-\xi z}J_0(\xi r)\,d\xi + \int_0^\infty B(\xi)K_0(\xi r)\sin(\xi z)\,d\xi + \int_0^\infty C(\xi)I_0(\xi r)\sin(\xi z)\,d\xi \tag{5}$$

where $J_0(x)$, $I_0(x)$, $K_0(x)$ are respectively, Bessel functions of the first kind and modified Bessel functions of the first and second kind.

It is convenient to write the integral transform $A(\xi)$ in terms of the finite integral given next;

$$A(\xi) = -\int_1^\gamma t\varphi(t)J_1(\xi t)\,dt \tag{6}$$

$$\varphi(r) = \partial\psi/\partial r \quad 1 \le r \le \gamma \tag{7a}$$

$$= 0 \quad \gamma \le r \le \beta. \tag{7b}$$

The choice for $A(\xi)$ given in eqn (6) automatically satisfies (3).

By noting that [3, p. 49(10)]

$$\int_0^\infty \xi(\xi^2 + \eta^2)^{-1}J_\nu(\xi r)J_\nu(\xi t)\,d\xi = I_\nu(\eta r)K_\nu(\eta t) \quad 0 < r < t \tag{8a}$$

$$= I_\nu(\eta t)K_\nu(\eta r) \quad 0 < t < r, \tag{8b}$$

applying boundary conditions (1), and taking Fourier sine transforms leads to the determination of $B(\xi)$ and $C(\xi)$ as

$$\frac{\pi}{2}\Delta B(\eta) = -I_1(\eta)I_1(\eta\beta)\int_1^\gamma t\varphi(t)K_1(\eta t)\,dt + I_1(\eta)K_1(\eta\beta)\int_1^\gamma t\varphi(t)I_1(\eta t)\,dt \tag{9a}$$

$$\frac{\pi}{2}\Delta C(\eta) = K_1(\eta)K_1(\eta\beta)\int_1^\gamma t\varphi(t)I_1(\eta t)\,dt - K_1(\eta\beta)I_1(\eta)\int_1^\gamma t\varphi(t)K_1(\eta t)\,dt \tag{9b}$$

where

$$\Delta = -[K_1(\eta)I_1(\eta\beta) - K_1(\eta\beta)I_1(\eta)]. \tag{10}$$

Finally, boundary condition (2) gives the following equation

$$\int_1^\gamma t\varphi(t)\int_0^\infty \xi J_1(\xi t)J_0(\xi r)\,d\xi - \frac{2}{\pi}\int_1^\gamma t\varphi(t)\int_0^\infty \frac{\eta I_1(\eta)}{\Delta}[I_1(\eta\beta)K_1(\eta t) - K_1(\eta\beta)I_1(\eta t)]K_0(\eta r)\,d\eta\,dt$$
$$+\frac{2}{\pi}\int_1^\gamma t\varphi(t)\int_0^\infty \frac{\eta K_1(\eta\beta)}{\Delta}[K_1(\eta)I_1(\eta t) - I_1(\eta)K_1(\eta t)]I_0(\eta r)\,d\eta\,dt = -p(r), \quad 1 \le r \le \gamma. \tag{11}$$

The infinite integral in the first term in eqn (11) can be written in the following simpler form

$$\int_0^\infty \xi J_1(\xi t)J_0(\xi r)\,d\xi = -\frac{2}{\pi}\left[\frac{1}{r^2-t^2}E\left(\frac{r}{t}\right)\right], \quad r<t \tag{12a}$$

$$= -\frac{2}{\pi}\left[\frac{r}{t}\cdot\frac{1}{r^2-t^2}E\left(\frac{t}{r}\right)-\frac{1}{rt}K\left(\frac{t}{r}\right)\right], \quad r>t \tag{12b}$$

$$\equiv K_0(r,t) \tag{12c}$$

where K and E are complete elliptic integrals of the first and second kinds, respectively.

Thus

$$\int_1^\gamma t\varphi(t)[K_0(r,t)+K_1(r,t)+K_2(r,t)]\,dt = -p(r), \quad 1\le r\le\gamma \tag{13}$$

where

$$K_1(r,t)=\frac{1}{\pi}\int_0^\infty \frac{\eta I_1(\eta)}{\Delta}[K_1(\eta\beta)I_1(\eta t)-I_1(\eta\beta)K_1(\eta t)]K_0(\eta r)\,d\eta \tag{14a}$$

$$K_2(r,t)=\frac{2}{\pi}\int_0^\infty \frac{\eta K_1(\eta\beta)}{\Delta}[K_1(\eta)I_1(\eta t)-I_1(\eta)K_1(\eta t)]I_0(\eta r)\,d\eta. \tag{14b}$$

In general the integrals given by eqns (14) will converge provided that $1<r,\ t<\beta$; however, in the limiting cases $\gamma\to 1$ and $\gamma\to\beta$ the convergence may not be especially good. The convergence of the integrals can be enhanced by subtracting functions from the integrand which behave like the leading terms in its asymptotic expansion. This technique is described by Krylov [4] and some further examples applied to oscillating integrands are given by Rice [5]. Thus $K_1(r,t)$ and $K_2(r,t)$ are modified as follows:

$$\begin{aligned}K_1(r,t)=\frac{2}{\pi}\int_0^\infty &\left\{\frac{\eta I_1(\eta)}{\Delta}[K_1(\eta\beta)I_1(\eta t)-I_1(\eta\beta)K_1(\eta t)]K_0(\eta r)\right.\\ &\left.-(1/4rt)^{1/2}[e^{-\eta(r+t-2)}-\alpha_1(r,t)(e^{-\eta(r+t-2)}-e^{-2\gamma\eta})/\eta]\right\}d\eta\\ &+(1/\pi^2 rt)^{1/2}\{1/(r+t-2)-\alpha_1(r,t)\ln[2\gamma/(r+t-2)]\}\end{aligned} \tag{15}$$

where

$$\alpha_1(r,t)=\frac{1}{8}\left(6+\frac{1}{r}-\frac{3}{t}\right) \tag{16}$$

and

$$\begin{aligned}K_2(r,t)=\frac{2}{\pi}\int_0^\infty &\left\{\frac{\eta K_1(\eta\beta)}{\Delta}[K_1(\eta)I_1(\eta t)-I_1(\eta)K_1(\eta t)]I_0(\eta r)\right.\\ &\left.+(1/4rt)^{1/2}[e^{-\eta(2\beta-t-r)}+\alpha_2(r,t)(e^{-\eta(2\beta-r-t)}-e^{-2\beta\eta})/\eta]\right\}d\eta\\ &-(1/\pi^2 rt)^{1/2}\{1/(2\beta-r-t)+\alpha_2(r,t)\ln[2\beta/(2\beta-r-t)]\}\end{aligned} \tag{17a}$$

where

$$\alpha_2(r,t)=\frac{1}{8}\left(\frac{6}{\beta}+\frac{1}{r}-\frac{3}{t}\right). \tag{17b}$$

The kernels given in the form of eqns (15) and (17) will allow their numerical evaluation with good accuracy for the limiting cases mentioned. Note that the terms involving $e^{-2\gamma\eta}$ and $e^{-2\beta\eta}$, in (15) and (17a), have been introduced artificially so that the $1/\eta$ asymptotic terms could be used without causing the integrals to diverge. The exponential argument factors of 2γ and 2β were chosen so that the artificial terms have little effect on the integrand values for large η.

The physical quantity of interest in this case is the intensity factor, which is defined as follows:

$$K = -\lim_{r\to\gamma}(\gamma - r)^{1/2}\frac{\partial\psi}{\partial r} \tag{18}$$

or equivalently

$$K/a^{1/2} = -\lim_{t\to\gamma}\left(\frac{\gamma - t}{a}\right)^{1/2}\varphi(t). \tag{19}$$

3. AXIALLY SYMMETRIC TORSION PROBLEM

The potential theory problem considered in this section corresponds to that of axially symmetric torsion for a cylindrical shell having a circumferential crack. The displacements and stresses corresponding to axially symmetric torsion are given in Green and Zerna[6] in terms of one potential function as

$$u_\theta = -\partial\psi/\partial r \tag{20a}$$

$$\tau_{z\theta}/\mu = -\partial^2\psi/\partial r\partial z \tag{20b}$$

$$\tau_{r\theta}/\mu = r^2\partial(r^{-1}\partial\psi/2r)/\partial r \tag{20c}$$

where ψ is given by eqn (5). Let

$$\psi_1 = -\partial\psi/\partial r \tag{21}$$

then ψ_1 is of the form

$$\psi_1 = \int_0^\infty A_1(\xi)e^{-\xi z}J_1(\xi r)\,d\xi + \int_0^\infty B_1(\xi)K_1(\xi r)\sin(\xi z)\,d\xi + \int_0^\infty C_1(\xi)I_1(\xi r)\sin(\xi z)\,d\xi. \tag{22}$$

The geometry of Fig. 1 is applicable and the boundary conditions are

$$\tau_{r\theta}/\mu = \partial(r^{-1}\psi_1)/\partial r = 0 \quad r = 1, \beta, \quad 0 \le z < \infty \tag{23a,b}$$

$$\tau_{z\theta}/\mu = \partial\psi_1/\partial z = -p(r) \quad z = 0, \quad 1 \le r \le \gamma \tag{24a}$$

$$u_\theta = \psi_1 = 0 \quad z = 0, \quad \gamma \le r \le \beta. \tag{24b}$$

As in the previous section

$$A_1(\xi) = -\int_1^a t\varphi_1(t)J_2(\xi t)\,dt \tag{25}$$

$$\varphi_1(r) = r\frac{d}{dr}\left(\frac{1}{r}\psi_1\right) \quad 1 \le r \le \gamma \tag{26a}$$

$$= 0 \quad \gamma \le r \le \beta \tag{26b}$$

where the choice of A_1 will automatically satisfy (24b). Applying boundary conditions (23), taking the Fourier sine transforms of the resulting equations, and using eqns (8), one is led to simultaneous equations for B_1 and C_1 whose solution is

$$\frac{\pi}{2}\Delta_1 B_1(\eta) = -I_2(\eta)I_2(\beta\eta)\int_1^\gamma t\varphi(t)K_2(\eta t)\,dt + I_2(\eta)K_2(\beta\eta)\int_1^\gamma t\varphi(t)K_2(\eta t)\,dt \tag{27a}$$

$$\frac{\pi}{2}\Delta_1 C_1(\eta) = K_2(\eta)K_2(\beta\eta)\int_1^\gamma t\varphi(t)I_2(\eta t)\,dt - I_2(\eta)K_2(\beta\eta)\int_1^\gamma t\varphi(t)K_2(\eta t)\,dt \tag{27b}$$

where

$$\Delta_1 = -[K_2(\eta)I_2(\beta\eta) - I_2(\eta)K_2(\beta\eta)]. \tag{28}$$

Boundary condition (24a) gives the following equation:

$$
\begin{aligned}
&\int_1^\gamma t\varphi_1 \int_0^\infty \xi J_2(\xi t) J_1(\xi r)\,\mathrm{d}\xi\,\mathrm{d}t \\
&\quad + \frac{2}{\pi}\int_1^\gamma t\varphi_1(t)\int_0^\infty \frac{\eta I_2(\eta)}{\Delta_1}[K_2(\beta\eta)I_2(\eta t) - I_2(\beta\eta)K_2(\eta t)]K_1(\eta r)\,\mathrm{d}\eta\,\mathrm{d}t \\
&\quad + \frac{2}{\pi}\int_1^\gamma t\varphi_1(t)\int_0^\infty \frac{\eta K_2(\beta\eta)}{\Delta_1}[K_2(\eta)I_2(\eta t) - I_2(\eta)K_2(\eta t)]I_1(\eta r)\,\mathrm{d}\eta\,\mathrm{d}t \\
&\quad = -p(r), \quad 1 \le r \le \gamma.
\end{aligned} \tag{29}
$$

Following the method in Section 2, eqn (29) can be written as

$$
\int_1^\gamma t\varphi_1(t)[R_0(r,t) + R_1(r,t) + R_2(r,t)]\,\mathrm{d}t = -p(r), \quad 1 \le r \le \gamma \tag{30}
$$

where

$$
R_0(r,t) = \frac{4}{\pi t^2}\left[K\left(\frac{t}{r}\right) - E\left(\frac{t}{r}\right)\right] + \frac{2}{\pi}\frac{1}{t^2 - r^2}E\left(\frac{t}{r}\right), \quad t < r \tag{31a}
$$

$$
= \frac{4}{\pi r t}\left[K\left(\frac{r}{t}\right) - E\left(\frac{r}{t}\right)\right] + \frac{2}{\pi r t}\left[\frac{t^2}{t^2 - r^2}E\left(\frac{r}{t}\right) - K\left(\frac{r}{t}\right)\right], \quad t > r \tag{31b}
$$

$$
\begin{aligned}
R_1(r,t) = \frac{2}{\pi}\int_0^\infty &\left\{\frac{\eta I_2(\eta)}{\Delta_1}[K_2(\beta\eta)I_2(\eta t) - I_2(\beta\eta)K_2(\eta t)]K_1(\eta r)\right. \\
&\left. - (1/4rt)^{1/2}[\mathrm{e}^{-\eta(r+t-2)} - \beta_1(r,t)(\mathrm{e}^{-\eta(r+t-2)} - \mathrm{e}^{-2\gamma\eta})/\eta]\right\}\mathrm{d}\eta \\
&+ (1/\pi^2 rt)^{1/2}\{1/(r+t-2) - \beta_1(r,t)\ln[2\gamma/(r+t-2)]
\end{aligned} \tag{32}
$$

where

$$
\beta_1(r,t) = \frac{3}{8}\left(10 - \frac{5}{t} - \frac{1}{r}\right) \tag{33}
$$

and

$$
\begin{aligned}
R_2(r,t) = \frac{2}{\pi}\int_0^\infty &\left\{\frac{\eta K_2(\beta\eta)}{\Delta_1}[K_2(\eta)I_2(\eta t) - I_2(\eta)K_2(\eta t)]I_1(\eta r)\right. \\
&\left. + (1/4rt)^{1/2}[\mathrm{e}^{-\eta(2\beta-t-r)} + \beta_2(r,t)(\mathrm{e}^{-\eta(2\beta-t-r)} - \mathrm{e}^{-2\beta\eta})/\eta]\right\}\mathrm{d}\eta \\
&- (1/\pi^2 rt)^{1/2}\{1/(2\beta - r - t) + \beta_2(r,t)\ln[2\beta/(2\beta - r - t)]\}
\end{aligned} \tag{34a}
$$

where

$$
\beta_2(r,t) = \frac{3}{8}\left(\frac{10}{\beta} - \frac{5}{t} - \frac{1}{r}\right). \tag{34b}
$$

For this case the quantity of physical interest is the intensity factor which is given as

$$
K/a^{1/2} = -\lim_{t\to\gamma}\left(\frac{\gamma - t}{a}\right)^{1/2}\varphi(t). \tag{35}
$$

4. NUMERICAL ANALYSIS

The numerical analysis for the integral eqns (13) and (30) given respectively in Sections 2 and 3 proceeds in identical fashion. The technique is based upon the collocation scheme for the solution of the singular integral equations given by Erdogan, Gupta and Cook[2]. Thus, by choosing the following change of variables in eqn (13)

$$
r = 1 + a(s+1)/2, \quad t = 1 + a(\tau + 1)/2,
$$

and by writing $\varphi(t)$ in the form

$$\varphi(t) = \Phi(t)(t-1)^{1/2}/(\gamma - t)^{1/2}$$

or

$$\varphi(\tau) = \Phi(\tau)(1+\tau)^{1/2}/(1-\tau)^{1/2}, \tag{36}$$

the system of simultaneous equations given below is obtained:

$$\frac{2\pi}{2n+1}\sum_{i=1}^{n}(1+\tau_i)\left[\frac{a}{2}(\tau_i+1)+1\right]\Phi(\tau_i)\{K_0(s_k,\tau_i)+K_1(s_k,\tau_i)+K_2(s_k,\tau_i)\}$$
$$= -p\left(1+\frac{1}{2}a+\frac{1}{2}as_k\right), \quad k = 1, 2, \ldots, n \tag{37}$$

where

$$\tau_i = \cos\left[\frac{2i-1}{2n+1}\pi\right] \quad i = 1, 2, \ldots, n \tag{38a}$$

$$s_k = \cos\left(\frac{2k\pi}{2n+1}\right) \quad k = 1, 2, \ldots, n. \tag{38b}$$

This amounts to applying a Gaussian quadrature formula for approximating the integral of a function $f(\tau)$ with weight function $[(1+\tau)/(1-\tau)]^{1/2}$ on the interval $[-1, 1]$. Thus,

$$\int_{-1}^{1}\left(\frac{1+\tau}{1-\tau}\right)^{1/2} f(\tau)\,d\tau \doteq \frac{2\pi}{2n+1}\sum_{i=1}^{n}(1+\tau_i)f(\tau_i) \tag{39}$$

where the τ_i are given by (38a). For example, see formula 25.4.43 in Ref.[7]. The procedure for the transformation of eqn (30) into a system of simultaneous equations is identical to the above and will not be listed here.

The integral equations were solved on a high speed digital computer using several numerical integration schemes to be discussed below. The computer used was a CDC 6600 having a word length of approximately fourteen decimal digits. The results, given as intensity factors, are summarized in Table 1. For the potential theory problem (denoted by I), $p(r) = p$(constant). Intensity factors defined by eqn (19) are given for $d = 0.25$, 0.10 and several values of a/d. The results appear to approach the asymptotic limit of $1/\sqrt{2}$ as $a/d \to 0$. The intensity factors defined by eqn (35) for the torsion problem are given for the two cases: IIa, $p(r) = p$ and IIb, $p(r) = pr/\beta$. The results appear to approach the asymptotic limits of $1/\sqrt{2}$ and $1/\beta\sqrt{2}$ for cases IIa and IIb as $a/d \to 0$.

To gain some confidence in the numerical results obtained we have examined several quadrature schemes for approximating the semi-infinite integrals (15), (17), (32), and (34). We have also made some comparisons where the number of collocation points were varied as well as the number of asymptotic terms ($\eta \to \infty$) subtracted from the integrand. Typical behaviors are shown in Tables 2 and 3. The quadrature schemes tried consisted of a 15 point Gauss–Laguerre routine GL15, a 40 point Gauss–Laguerre routine GL40, and a semi-adaptive routine QINFI

Table 1. Intensity factors (I: $\lim_{t\to\gamma} -[(\gamma-t)/a]^{1/2}\varphi(t)/p$; IIa, IIb: $\lim_{t\to\gamma} -[(\gamma-t)/a]^{1/2}\varphi_1(t)/p$)

	a/d	.05	.2	.4	.6	.8	.95
(I)	d = .25	.7036	.7027	.7273	.8009	1.017	1.870
	d = .10	.7061	.7122	.7461	.8309	1.067	1.975
(IIa)	d = .25	.6983	.6834	.6904	.7442	.9291	1.695
	d = .10	.7039	.7038	.7292	.8041	1.024	1.890
(IIb)	d = .25	.5631	.5644	.5880	.6519	.8329	1.536
	d = .10	.6420	.6480	.6797	.7579	.9741	1.806

Table 2. Sample behavior of semi-infinite integral evaluation when subtracting off two, one or none of the leading terms in the asymptotic form of the integrand

	r = t = 1.000007			r = t = 1.0005		
	2-term	1-term	0-term	2-term	1-term	0-term
Asymptotic part	35701.2	35711.5	0.	41.7195	47.1328	0.
GL15	35705.8	35710.8	23.9	46.0839	46.46	17.6
GL40	35705.8	35710.0	71.5	46.0840	46.16	35.2
QINFI	35705.8	35705.8	35705.8	46.0840	46.0841	46.0841

Table 3. Sample behavior in problem solution $\Phi(\gamma)$ as we vary the number of collocation points using a 2-term asymptotic formulation and then varying the number of asymptotic terms being subtracted

	Potential Problem		Torsion Problem	
	a = 0.005 d = 0.1	a = 0.2375 d = 0.25	a = 0.005 d = 0.1	a = 0.2375 d = 0.25
10 points	0.7062	1.873	0.7041	1.708
20 points	0.7061	1.870	0.7040	1.699
30 points	0.7061	1.869	0.7039	1.695
1-term	0.7057	1.870	0.7026	1.694
0-term	0.895	1.788	0.8898	1.621

which examines convergence behavior in order to achieve the requested error tolerance. Typically, QINFI requires from 3 to 10 times the cost of using GL40 while GL40 requires about 2.5 times the cost of using GL15.

Table 2 shows the results of each of these routines when two, one or none of the leading terms in the asymptotic form of the integrand are subtracted before integration. The problem being examined corresponds to $a = 0.005$, $d = 0.1$ and the use of 20 collocation points. The corresponding values of r and t represent (approximately) the two extremes actually used in the computation of the collocation matrix for determining the solution of the integral equation. We conclude that it is safe to use the less accurate and least expensive routine GL15 provided we subtract off the leading two terms in the asymptotic expansion of the integrand as $\eta \to \infty$.

Table 3 shows the behavior in convergence of the collocation scheme for several problems of interest. This examination led us to use 20 points for the potential problems and 30 points for the torsion problems. Thus, using GL15 and a two term asymptotic subtraction scheme the results shown in Table 1 were obtained. We believe the values to be accurate to about three significant figures. Typical computation times were about 14 sec for the potential theory problem and 78 seconds for the torsion problem.

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